Chi-Square and EDF tests for student's T distribution using selective order statistics

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Abstract

This study aims to assess the effectiveness of empirical distribution function (EDF) tests and the chi-square test for goodness of fit (GOF) when applied to Student's t distribution under Selective Order Statistics(SOS). Through a simulation process, we compare the power and efficiency of these tests under SOS and simple random sampling (SRS). Results indicate that EDF tests and the chi-square test exhibit greater power under SOS than SRS. However, in the case of median SOS, the chi-square test is found to be more powerful under SRS. Additionally, we provide the percentage points of these tests under null hypotheses.

Keywords: Chi-square test, Goodness of fit test; Order statistics; Power; Student's t distribution.

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اختبار كاي واختبارات حسن التطابق لتوزيع T باستخدام بعض الاحصاءات المرتبة

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ملخص

ساهمت هذه الدراسة بتقييم فعالية اختبارات حسن التطابق واختبار كاي على توزيع T مستخدما بعض الاحصاءات المرتبة. باستخدام طريقة المحاكاة، تم مقارنة هذه الاختبارات لمجموعه من التوزيعات المتماثلة والملتوية بطريقتين وهما العينات المرتبة والعينات البسيطة. اضهرت النتائج ان هذه الاختبارت أكثر فعالية عند استخدام العينات المرتبة باستثناء حالة الوسيط مقارنة مع طريقة العينات البسيطة. ايضا تم حساب النقاط المئوية لهؤلاء الاختبارات تحت النظرية الصفرية.

الكلمات المفتاحية: اختبار كاي، اختبار حسن التطابق، الاحصاءات المرتبة، قوة الاختبار، توزيع تي.

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Introduction:

In the field of probability and statistics, Student's t distribution is part of a continuum of probability distributions. It comes into play when estimating the mean of a population that follows a normal distribution, especially in scenarios with small sample sizes and unknown population standard deviations. William Sealy Gosset formulated this distribution using the alias "Student." Student's t distribution exhibits similarities to the normal distribution, such as its symmetry and bell-shaped curve. However, it diverges by having heavier tails and a somewhat broader shape. This unique trait makes it invaluable for scrutinizing the statistical tendencies of specific ratios of random variables. In instances where the variability in the denominator is magnified and can lead to outlier values, particularly as the denominator approaches zero, Student's t distribution proves particularly beneficial (Li et al., 2018).

In this paper, we assume that the sample is selected using the selective order statistics method. Subsequently, we employ this method to establish empirical distribution function (EDF) and chi-square goodness of fit (GOF) tests tailored for the Student's t distribution. Moreover, the content of this paper is part of the relatively new work on goodness of fit. The novelty of this study lies in the application of testing a student’s t distribution, which looks a more advanced scheme than the usual normal distribution study. Traditional goodness of fit tests often treat all data points equally, which may not capture the specific characteristics of the dataset. By selectively focusing on certain subsets of the data, such as extreme values or values within particular ranges, these tests allow for a more nuanced evaluation of how well the t-distribution fits the observed data. Another contribution is the enhanced robustness of these tests to deviations from normality in the data. The t-distribution is often used as an alternative to the normal distribution when dealing with small sample sizes or when the assumption of normality is questionable. Chi-square and EDF tests using selective order statistics provide a means to assess the appropriateness of the t-distribution assumption even in cases where the data may not follow a perfect normal distribution.

The fundamental concept of choosing a sample of size m*r through the Ranked Set Sampling (RSS) procedure can be outlined as follows: Firstly, select m independent simple random samples from the population of interest, each of size m. Next, through visual inspection or an economical method, rank the units within each sample without quantification based on the variable of interest. Subsequently, choose the ith smallest unit from each sample for actual measurement, i=1,2,..,m. This process results in a total of m measured units, one from each sample. The illustration of the RSS procedure is presented below, with m representing the independent simple random samples.

Table (1) The *m* independent SRS

Sample	SRS
1	$\{X_{11} \quad X_{12} \quad \cdots \quad X_{1m}\}$
2	$\{X_{21} \quad X_{22} \quad \cdots \quad X_{2m}\}$
$\vdots$	$\vdots \quad \vdots \quad \ddots \quad \vdots$
m	$\{X_{m1} \quad X_{m1} \quad \cdots \quad X_{mm}\}$

Then, if each row is ranked visually, we obtain *m* ordered samples which are given in Table 2.

Table (2) Ordered samples

Sample	Ordered SRS
1	$\{X_{1(1)} \quad X_{1(2)} \quad \cdots \quad X_{1(m)}\}$
2	$\{X_{2(1)} \quad X_{2(2)} \quad \cdots \quad X_{2(m)}\}$
$\vdots$	$\vdots \quad \vdots \quad \ddots \quad \vdots$
m	$\{X_{m(1)} \quad X_{m(1)} \quad \cdots \quad X_{m(m)}\}$

where $X_{i(j)}$ denotes the j^{th} order statistics of the i^{th} set.

1. If the diagonal elements $\{X_{1(1)}, X_{2(2)}, \cdots, x_{m(m)}\}$ are chosen for actual measurements, then they represent an RSS of size *m*. The procedure could be repeated *r* times until a sample of $n = m * r$ measurements are obtained. These mr measurements form an RSS of size mr.
2. The term for the resulting sample, when only the median of each set is chosen for actual quantification, is known as Median Ranked Set Sampling (MRSS).
3. When only the i^{th} order statistic of each set is chosen for quantification, the resulting sample is termed Selective Order Statistic.

Regarding its performance as an estimator of the population mean, (McIntyre, 1952) demonstrated that the mean in RSS serves as an unbiased estimator, proving to be more efficient than the sample mean derived from a SRS of equivalent size.

The foundational principles of RSS were initially outlined by Stockes et al. (1988), elucidating that the mean of an RSS serves as the minimum variance unbiased estimator for the population mean. Dell et al. (1972) further affirmed that the RSS mean remains unbiased and demonstrates superior efficiency compared to the SRS mean, even under imperfect ranking. Muttlak (1997) introduced the concept of Median RSS (MRSS), which focuses solely on quantifying the median in each set. Another iteration, Varied Set Size RSS, later termed Moving Extremes RSS (MERSS), was proposed by Al-Odat et al. (2001), investigating its applicability to the location-scale family and revealing its capacity to yield more efficient estimators for location and scale parameters. Samawi et al. (1996) explored the estimation of the distribution function by examining extreme and median RSS. For a comprehensive overview of RSS developments, references such as Samawi et al. (2001), Muttlak (2003) and Al-Subh (2018) provide valuable insights.

A comprehensive examination of GOF tests relying on SRS is documented in the book by D'Agostino et al. (1986). Stephens (1974) conducted extensive research into the characterization of RSS, presenting an unbiased estimator for the population distribution function utilizing the empirical distribution function (EDF) of RSS. Additionally, they proposed a Kolmogorov-Smirnov GOF test based on the EDF and derived its null distribution. Ibrahim et al. (2011) introduced an innovative method to enhance the power of the chi-square test for goodness of fit based on RSS, employing a simulation study to evaluate the effectiveness of this new approach.

The paper is structured as follows: Section 2 introduces a method aimed at enhancing the EDF and chi-square test statistics for GOF, and we specifically apply this method to the Student's t distribution. Following this, an algorithm is formulated to compute the power function and efficiency under an alternative distribution. Finally, a simulation study is executed to compare the power of the EDF and chi-square test statistics under the selective order statistics (RSS) method with their counterparts in simple random sampling (SRS). Section 3 presents the simulation results, and in Section 4, we provide our conclusions based on the findings of the study.

Material and Methods

1. Chi-square test for goodness of fit

The fundamental concept behind chi-squared tests involves simplifying the general fitting test by transforming it into a comparison between observed cell counts and their expected values under the hypothesis being tested. While chi-squared tests are broadly applicable, they can sometimes be less powerful than other tests. This reduced power is attributed to the information loss resulting from the grouping of data.

We adopt the assumption that the set size in McIntyre's RSS is odd. This decision simplifies computations when comparing our approach with median RSS. Should the set size be even in certain scenarios, our theoretical framework can be smoothly expanded. Let $X_1, X_2, ..., X_r$ be a random sample from the distribution function *F(x)*. The goal is to test the statistical hypotheses

$$H_o : F(x) = F_o(x) \quad \forall x, \text{ vs. } H_1 : F(x) \neq F_o(x) \qquad (1)$$

for some x, where $F_o(x)$ is a known distribution function. The chi-square χ^2 test statistic is studied for GOF which can be described as follows. Let $I_1, I_2, ..., I_{k+1}$ be a partition of the support of $F_o(x)$ and N_j = number of X_i 's that fall in I_j, $j = 1, 2, ..., k+1$. For large n, the hypothesis (2.1) is rejected if

$$\chi^2 = \sum_{j=1}^{k+1} \frac{(N_j - nP_j)^2}{nP_j} > \chi^2_{1-\alpha,\, k}$$

where $P_j = P_{F_o}(X_i \in I_j)$, $j = 1, 2, ..., k+1$ and $\chi^2_{1-\alpha,\, k}$, is the $(1-\alpha)100^{th}$ quantile of the chi-square distribution with k degrees of freedom.

Continuing with the discussion, we now introduce the chi-square test under selective order RSS. We maintain the assumption that the set size in McIntyre's RSS is odd, i.e., 2m-1. This choice ensures straightforward calculations when comparing the method with MRSS. If the set size were even, the theory developed here could be extended seamlessly. In this study, our focus is on exploring the performance of the test, particularly when m=2, corresponding to the selective order RSS based on minimum, median, and maximum. Notably, testing the hypothesis in (1) is equivalent to testing the hypothesis

$H_o^*: F_{i:n}(x) = F_{i:n(o)}(x) \quad \forall x \quad$ vs. $\quad H_1^*: F_{i:n}(x) \neq F_{i:n(o)}(x),$ (2)

for some x, where $F_{i:n}(x)$ and $F_{i:n(o)}(x)$ are the cumulative distribution functions (cdfs) of the i^{th} order statistics where $i = 1, \ldots, 2m-1,$ chosen from $F(x)$ and $F_o(x)$ respectively. According to Arnold, et al. (1992), $F_{i:n}(x)$ and $F_{i:n(o)}(x)$ have the following representations:

$$F_{i:n}(x) = \sum_{j=i}^{2m-1} \binom{2m-1}{j} [F(x)]^j [1 - F(x)]^{(2m-1)-j}, \quad (3)$$

and

$$F_{i:n(o)}(x) = \sum_{j=i}^{2m-1} \binom{2m-1}{j} [F_o(x)]^j [1 - F_o(x)]^{(2m-1)-j}, \quad (4)$$

respectively.

In the case of $m = 2,$ we have $i = 1,$ 2 and 3. Thus, for different values of I the cdfs $F_{i:n}(x)$ and $F_{i:n(o)}(x)$ are given by

$$F_{1:n}(x) = F^3(x) - 3F^2(x) + 3F(x) = 1 - (1 - F(x))^3,$$

$$F_{1:n(o)}(x) = 1 - (1 - F_o(x))^3,$$

$$F_{2:n}(x) = 3F^2(x)(1 - F(x)) + F^3(x) \quad \text{and}$$

$$= 3F^2(x) - 2F^3(x),$$

$$F_{2:n(o)}(x) = 3F_o^2(x) - 2F_o^3(x),$$

$$F_{3:n}(x) = F^3(x),$$

$$F_{3:n(o)}(x) = F_o^3(x).$$

It can be shown that the equation $F_{i:n}(x) = F_{i:n(o)}(x)$ has the unique solution $F(x) = F_o(x).$

To define the chi-square test statistic under selective order statistic, consider a random sample of size n, which is a copy of the i^{th} order statistics, denoted as $X_{(i:m)1}, X_{(i:m)2}, \ldots, X_{(i:m)n}$. Let I_j, $j = 1, \ldots, k+1$ be a partition of $(-\infty, \infty)$. Let M_j = number of $X_{(i:m)j}$'s that falls in the interval I_j where $j = 1, \ldots, k+1.$ The chi-square test statistic applied under RSS for testing the hypotheses H_o^* vs. H_1^*, is formulated as follows

$$\chi^{*2} = \sum_{j=1}^{k+1} \frac{(M_j - n\,P_j^*)^2}{n\,P_j^*}, \quad (5)$$

where $P_j^* = \int_{I_j} dF_{i:n(o)}(x).$

The hypothesis H_o^* is rejected at level α if $\chi^{*2} > \chi^2_{1-\alpha,\, k}$. The power of the χ^{*2} test, denoted by $\pi(\chi^{*2}),$ can be calculated based on the probability equation

$$\pi(\chi^{*2}) = P_H(\chi^{*2} > \chi^2_{1-\alpha,\, k}), \qquad (6)$$

where H is a cdf under the alternative hypothesis H_1^*.

2. EDF tests for goodness of fit

Stephens (1974) provided a practical guide to goodness of fit tests using statistics based on the empirical distribution function (EDF). In his work, he compared the following EDF tests

a) The Kolmogorov statistics: $D^+,\ D^-,\ D$

$$D^+ = \max_{1 \le i \le n} \left[(i/n) - z_i \right].$$

$$D^- = \max_{1 \le i \le n} \left[z_i - (i-1)/n \right].$$

$$D = \max \left[D^+, D^- \right].$$

b) The Cramer-von Mises statistics: W^2

$$W^2 = \sum_{i=1}^{n} \left[z_i - (2i-1)/2n \right]^2 + (1/12n).$$

c) The Kuiper statistic: V

$$V = D^+ + D^-.$$

d) The Watson statistics: U^2

$$U^2 = W^2 - n\left(\bar{z} - \frac{1}{2} \right)^2 \qquad \text{where} \qquad \bar{z} = \sum_{i=1}^{n} z_i / n.$$

e) The Anderson-Darling statistic: A^2

$$A^2 = -\left\{ \sum_{i=1}^{n} (2i-1)\left[\ln z_i + \ln(1 - z_{n+1-i}) \right] \right\} / n - n. \qquad (7)$$

Testing for Student's t distribution

1. Chi-Square Test

For a random variable X following a Student's t distribution with r degrees of freedom, the probability density function (pdf) and cumulative distribution function (cdf) are given by:

$$f(x) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)}\left(1+\frac{x^2}{r}\right)^{-\frac{r+1}{2}}, \qquad (8)$$

$$F(x) = 0.5 + \frac{x\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{r\pi}\,\Gamma\left(\frac{r}{2}\right)}\ {}_2F_1\left(0.5, \frac{r+1}{2}, 1.5, \frac{-x^2}{r}\right), \qquad (9)$$

Where $r > 0$ *and* ${}_2F_1$ denotes the Gauss hypergeometric function, denoted as $X{:}\,S(r)$.

Testing the hypothesis in (1) is equivalent to testing the hypothesis in (2). To facilitate this comparison, consider the partition of $(-\infty,\infty)$ such that

$$I_1 = (-\infty, a],\ where\ \ a \in R, I_j = \left((j-1)a, ja\right], for\ j = 2,3,\dots,k \quad and\ \ I_{k+1} = (ka,\infty). \qquad (10)$$

Let M_j = number of X_i's that falls in the interval I_j, $j = 1,\dots, k+1$.

Thus, for $j = 2,\dots,k$ we have

$$P_j^* = \int_{(j-1)a}^{ja} dF_{i:n(o)}(x) = F_{i:n(o)}(ja) - F_{i:n(o)}((j-1)a).$$

For $j = 1$ and $k+1$, we have $P_1^* = F_{i:n(o)}(a)$, and $P_{k+1}^* = 1 - F_{i:n(o)}(ka)$ (11)

respectively. So, we reject H_o^* at level of significance α if

$$\chi^{*2} = \sum_{j=1}^{k+1}\frac{(M_j - n\,P_j^*)^2}{n\,P_j^*} > \chi^2_{1-\alpha,\,k}. \qquad (12)$$

2. EDF Tests

If we utilize RSS to collect the data using the i^{th} order statistic, we can subsequently employ the resulting data to construct empirical distribution function goodness of fit tests for the hypotheses in equation (2). Let $Y_1,\dots,Y_r$ be a random sample of size *r* selected via the i^{th} order statistic. Let T_1 denote a test in equation (7), and T_1^* denotes its counterpart in the RSS when testing (2) using the data $Y_1,\dots,Y_r$.

In this paper, we focus our analysis on the case when $F_o(x)$, specifically for the Student's T_1 distribution. Furthermore, we conduct a simulation study to demonstrate that the test T_1^* exhibits greater statistical power compared to the test T_1 when both are assessed based on samples of equal size. The power of the T_1^* test can be calculated according to the equation

$$\text{Power of } T_1^*(H) = P_H(T_1^* > d_\alpha) \qquad (13)$$

where H represents the cumulative distribution function (cdf) under the alternative hypothesis H_1^*. Here, 100α denotes the percentage point of the distribution of T_1^* and d_α is the corresponding probability level of H_o. Given the comparison of RSS test statistics to SRS test statistics, the efficiency of the test statistics will be calculated as the ratio of powers.

$$eff(T_1^*, T_1) = \frac{power\ of\ T_1^*}{power\ of\ T_1} \qquad (14)$$

T_1^* is more powerful than T_1 if $eff(T_1^*, T_1) > 1$.

Power comparison

1. Chi-Square test

The comparison of the performance between χ^2 and χ^{*2} is conducted by evaluating the power and efficiency of the test under SRS and RSS for samples of the same size. To calculate the power of the test under RSS, the following algorithm is designed

Step 1: Select a sample of size n from H, a distribution under the alternative hypothesis H_1.

Step 2: Classify the sample obtained in step 1 into the $k+1$ subintervals $I_1, I_2, ..., I_{k+1}$, as given in (9), to obtain the frequencies $M_1, M_2, ..., M_{k+1}$.

Step 3: Obtain the values of $P_1^*, P_2^*, ..., P_{k+1}^*$ as follows:

$P_1^* = F_{i:n(o)}(a),\ P_j^* = F_{i:n(o)}(ja) - F_{i:n(o)}((j-1)a),\ i = 2, ..., k$, and

$P_{k+1}^* = 1 - F_{i:n(o)}(ka)$.

Step 4: Calculate χ^{*2} from equation (12).

Step 5: Repeat the steps (1) to (4), 10,000 times to get $\chi_1^{*2}, ..., \chi_{10,000}^{*2}$.

Step 6: Approximate the power of the χ^{*2} test under H as follows

$$\pi(\chi^{*2}) = P_H(\chi^{*2} > \chi^2_{1-\alpha^*, k}) \approx \frac{1}{10,000} \sum_{t=1}^{10,000} I(\chi_t^{*2} > \chi^2_{1-\alpha, k}),$$

where $I(.)$ stands for the indicator function. To calculate the power of the test under SRS, a similar algorithm to that of RSS is designed, but it involves data generated under SRS. The efficiency is then calculated using the following ratio:

$$eff(\mathrm{T}_2^*, T_2) = \frac{\Pi(\chi^{*2})}{\Pi(\chi^2)} \qquad (15)$$

2. EDF tests

To compare the powers of T_1^* *and* T_1 , we initially devise the following algorithm to determine the critical values:

1. Simulate $Y_1,...,Y_r$ be a RSS obtained based on the i^{th} order statistic from $G_{i0}(x),\ i=1,\ 2,\ 3.$
2. Without loss of generality we assume $\theta=0,\ \sigma=1$.
3. Find the EDF $F_r^*(x)$ as follows:

$$F_r^*(x) = \frac{1}{r}\sum_{j=1}^{r} I(Y_{(i)j} \le x),\quad I(Y_{(i)j} \le x) = \begin{cases} 1 & ,Y_{(i)j} \le x, \\ 0 & , \quad \text{o.w,} \end{cases} \qquad (16)$$

4. Use $F_r^*(x)$ to calculate the value of T_1^* as in (13).
5. Repeat the steps (1-4) 10, 000 times to get $\mathrm{T}_{1,1}^*, \dots, \mathrm{T}_{1,10000}^*$.
6. The critical value d_α of T_1^* is given by the $(1-\alpha)100\%$ quantile of $\mathrm{T}_{1,1}^*, \dots, \mathrm{T}_{1,10000}^*$.

Next, for the computation of the power of T_1^* at H, simulation is employed. Hence, the following algorithm is designed:

1. Simulate $Y_1,...,Y_r$ be a RSS obtained based on the i^{th} order statistic from H, a distribution under H_1^*, $i=1,\ 2,\ 3.$
2. Find the EDF $F_r^*(x)$ as in (16).
3. Calculate the value of T_1^* as in (2.1) but using the data $Y_1,...,Y_r$.
4. Repeat the steps (1) - (3), 10,000 times to get $\mathrm{T}_{1,1}^*, \dots, \mathrm{T}_{1,10000}^*$. .
5. Power of $\mathrm{T}_1^*(H) \approx \frac{1}{10000}\sum_{t=1}^{10000} I(\mathrm{T}_{1t}^* > d_\alpha)$, where $I(.)$ stands for indicator function.

Results and Discussion

1. Chi-Square test

In comparing the test statistic under RSS against SRS, we explore five symmetric distributions—namely, normal, logistic, Cauchy, Laplace, and uniform—and two asymmetric distributions—lognormal and exponential. Conducting a Monte Carlo simulation with 10,000 iterations, we calculate the power of each test statistic. Subsequently, we compute and compare the powers of the two tests across various sample sizes, i.e. $n = 20, 30, 40$, different set sizes, i.e. $2m-1$, where $m = 1, 2, 3, 4$ (m=1 refers to SRS case), different number of intervals, i.e. $k = 5, 10, 15$, and different alternative distributions, i.e.

$Normal(0,1) = N, Logistic(0,1) = Lo, Lognormal\ (0,1) = LN, Cauchy(0,1) = C,$ $Laplace(0,1) = Lp,\quad Uniform(0,1) = U\ and\ Exponential\ (1) = E.$

Comparisons are exclusively performed for scenarios where the data is quantified using the minimum, maximum, and median. The power values of the tests are documented in Tables 3 and 4, respectively. The efficiencies of the tests are outlined in Tables 5 and 6, respectively. In the case of the maximum, the tables are omitted.

Table (3) The power of the chi-square tests under Simple Random Sampling (SRS) and Ranked Set Sampling (RSS) based on the first order statistic is evaluated for various alternative distributions and values of m, k, n and

H	**k=5,Min ,** $\alpha = 0.05$			**k=10**			**k=15**		
	$m = 1$						r		
	20	30	40	20	30	40	20	30	40
N	.003	.007	.010	.008	.011	.014	.009	.013	.019
Lo	.551	.614	.670	.608	.693	.749	.641	.702	.782
C	.911	.962	1	.911	.965	1	.913	.970	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	.177	.171	.185	.195	.215	.230	.231	.225	.249
	$m = 2$						r		
	20	30	40	20	30	40	20	30	40
N	.006	.012	.019	.016	.028	.035	.011	.015	.025
Lo	.398	.536	.645	.470	.622	.717	.456	.582	.680
C	.915	.965	1	.915	.970	1	.916	.970	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1

H	**k=5,Min** , $\alpha = 0.05$			**k=10**			**k=15**		
Lp	.078	.092	.103	.101	.152	.165	.088	.120	.172
	$m = 3$						r		
	20	30	40	20	30	40	20	30	40
N	.009	.018	.022	.013	.031	.042	.035	.042	.055
Lo	.490	.635	.769	.441	.593	.682	.411	.551	.645
C	1	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	.092	.095	.105	.062	.085	.103	.077	.080	.082
	$m = 4$						r		
	20	30	40	20	30	40	20	30	40
N	.017	.040	.055	.015	.035	.045	.012	.024	.035
Lo	.563	.821	.951	.447	.612	.692	.365	.521	.631
C	1	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	.077	.099	.120	.066	.077	.092	.045	.048	.055

Table (4) The power of the chi-square tests under Simple Random Sampling (SRS) and Ranked Set Sampling (RSS) based on the median order statistic is evaluated for various alternative distributions and values of m, k, n and $\alpha = 0.05$

H	k=5,Median , $\alpha = 0.05$			k=10			k=15		
	$m = 1$					r			
	20	30	40	20	30	40	20	30	40
N	.007	.010	.012	.008	.012	.015	.009	.015	.019
Lo	.552	.605	.661	.612	.683	.752	.643	.715	.782
C	.911	1	1	.911	1	1	.911	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	.177	.172	.185	.195	.215	.230	.231	.225	.249
	$m = 2$						r		
	20	30	40	20	30	40	20	30	40
N	.836	.967	1	.919	1	1	.951	1	1
Lo	1	1	1	1	1	1	1	1	1
C	1	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	1	1	1	1	1	1	1	1	1
	$m = 3$						r		
	20	30	40	20	30	40	20	30	40

H	k=5,Median , $\alpha = 0.05$			k=10			k=15		
N	1	1	1	1	1	1	1	1	1
Lo	1	1	1	1	1	1	1	1	1
C	1	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	1	1	1	1	1	1	1	1	1
	$m = 4$						r		
	20	30	40	20	30	40	20	30	40
N	1	1	1	1	1	1	1	1	1
Lo	1	1	1	1	1	1	1	1	1
C	1	1	1	1	1	1	1	1	1
U	1	1	1	1	1	1	1	1	1
LN	1	1	1	1	1	1	1	1	1
E	1	1	1	1	1	1	1	1	1
Lp	1	1	1	1	1	1	1	1	1

Table (5) The efficiency of the chi-square test under Selective Order Ranked Set Sampling (RSS) compared to Simple Random Sampling (SRS) based on minimum order statistics is assessed for various alternative distributions and values of m, k, n and $\alpha = 0.05$

H		k=5, Min , $\alpha = 0.05$			k=10			k=15		
						r				
	m	20	30	40	20	30	40	20	30	40
N	2	2	1.7143	1.9	2	2.545	2.5	1.222	1.154	1.316
Lo	2	0.722	0.873	0.963	0.773	0.898	0.957	0.711	0.829	0.870
C	2	1.004	1.0031	1	1.004	1.005	1	1.003	1	1
U	2	1	1	1	1	1	1	1	1	1
LN	2	1	1	1	1	1	1	1	1	1
E	2	1	1	1	1	1	1	1	1	1
Lp	2	0.441	0.538	0.557	0.518	0.707	0.717	0.381	0.533	0.691
						r				
		20	30	40	20	30	40	20	30	40
N	3	2	1.7143	1.9	2	2.545	2.5	1.222	1.154	1.316
Lo	3	0.722	0.873	0.963	0.773	0.898	0.957	0.711	0.829	0.870
C	3	1.004	1.0031	1	1.004	1.005	1	1.003	1	1
U	3	1	1	1	1	1	1	1	1	1
LN	3	1	1	1	1	1	1	1	1	1
E	3	1	1	1	1	1	1	1	1	1
Lp	3	0.441	0.538	0.557	0.518	0.707	0.717	0.381	0.533	0.691
						r				
		20	30	40	20	30	40	20	30	40

H		k=5, Min , $\alpha = 0.05$			k=10			k=15		
N	4	3	2.5714	2.2	1.625	2.818	3	3.889	3.231	2.895
Lo	4	0.889	1.0342	1.148	0.725	0.856	0.911	0.641	0.785	0.825
C	4	1.098	1.0395	1	1.098	1.036	1	1.095	1.031	1
U	4	1	1	1	1	1	1	1	1	1
LN	4	1	1	1	1	1	1	1	1	1
E	4	1	1	1	1	1	1	1	1	1
Lp	4	0.52	0.5556	0.568	0.318	0.395	0.448	0.333	0.356	0.329

Table (6) The efficiency of the chi-square test under Selective Order Ranked Set Sampling (RSS) compared to Simple Random Sampling (SRS) based on median order statistics is assessed for various alternative distributions and values of m, k, n and $\alpha = 0.05$

H		k=5, Med , $\alpha = 0.05$			k=10			k=15		
					r					
	m	20	30	40	20	30	40	20	30	40
N	2	119	96.7	83.3	115	83.3	66.7	106	66.7	52.6
Lo	2	1.81	1.653	1.51	1.63	1.46	1.33	1.56	1.4	1.28
C	2	1.1	1	1	1.1	1	1	1.1	1	1
U	2	1	1	1	1	1	1	1	1	1
LN	2	1	1	1	1	1	1	1	1	1
E	2	1	1	1	1	1	1	1	1	1
Lp	2	5.65	5.814	5.41	5.13	4.65	4.35	4.33	4.44	4.02
					r					
		20	30	40	20	30	40	20	30	40
N	3	143	100	83.3	125	83.3	66.7	111	66.7	52.6
Lo	3	1.81	1.653	1.51	1.63	1.46	1.33	1.56	1.4	1.28
C	3	1.1	1	1	1.1	1	1	1.1	1	1
U	3	1	1	1	1	1	1	1	1	1
LN	3	1	1	1	1	1	1	1	1	1
E	3	1	1	1	1	1	1	1	1	1
Lp	3	5.65	5.814	5.41	5.13	4.65	4.35	4.33	4.44	4.02
					r					
		20	30	40	20	30	40	20	30	40
N	4	143	100	83.3	125	83.3	66.7	111	66.7	52.6
Lo	4	1.81	1.653	1.51	1.63	1.46	1.33	1.56	1.4	1.28
C	4	1.1	1	1	1.1	1	1	1.1	1	1
U	4	1	1	1	1	1	1	1	1	1
LN	4	1	1	1	1	1	1	1	1	1
E	4	1	1	1	1	1	1	1	1	1
Lp	4	5.65	5.814	5.41	5.13	4.65	4.35	4.33	4.44	4.02

Based on Tables 3 to 6, the following conclusions can be drawn for symmetric distributions

1. When comparing minimum and maximum order statistics, the power increases with an increase in either sample size n or set size m.
2. For both minimum and maximum order statistics, the power increases as the number of intervals k increases from *k=5 to k=15*.
3. Regarding median order statistics, the power increases with an increase in sample size n but decreases with an increase in set size m.
4. For both minimum and maximum order statistics, the efficiency decreases as sample size n increases.
5. For both minimum and maximum order statistics, the efficiency increases as set size m increases.
6. The chi-square test is more efficient for minimum and maximum order statistics but not efficient for median order statistics when comparing the same sample sizes.
7. It is evident from Table 6 that the efficiencies of the tests for median order statistics are greater than or equal to one.
8. When the uniform distribution is chosen under the alternative hypothesis, the power of the chi-square tests is close to one.

For asymmetric distributions, the following observations are made:

1. The power increases with an increase in both sample size n and set size m.
2. When comparing samples of the same size, the chi-square tests based on minimum and maximum order statistics are found to be more powerful than their counterparts in SRS for the considered distributions.
3. For all order statistics, the powers are all equal to one for lognormal and exponential distributions.
4. The power is low for a large number of intervals.
5. Regarding median order statistics, the power increases with an increase in sample size n but decreases with an increase in set size m.

6. The chi-square test is more efficient for minimum and maximum order statistics but not efficient for median order statistics for the same sample size.
7. When the exponential and lognormal distributions are chosen under the alternative hypothesis, the power of the chi-square tests is approximately one.

2. EDF Tests

We estimated the power of each test using a Monte Carlo simulation comprising 10,000 iterations, following the algorithm described in Section 2. Table 7 presents the percentage points for the 5-percent level for the null hypotheses of Student's T distribution with 5 degrees of freedom under three sampling schemes: first(*i=1*), second (*i=2*), and largest (*i=3*) order statistics. The powers and efficiencies of the two tests were compared across various sample sizes: $r = 10, 20, 30, 40$, different set sizes (*m=1* representing the mean SRS case), and different alternative distributions: $Normal(0,1) = N, Logistic(0,1) = Lo, Lognormal\ (0,1) = LN, Cauchy(0,1) = C,$ $Laplace(0,1) = La, \quad Uniform(0,1) = U\ and\ Expoential(1) = E.$

Simulation results are presented in Tables (7)-(9). For Lognormal and Uniform alternative distributions, the computations reveal that the powers and efficiencies of all test statistics are equal to one. Therefore, these values have not been reported in Tables (7) and (9). The tables related to the maximum are excluded for the symmetric distribution.

Table (7) The Percentage points for the first, median and largest order statistics

	Minimum, $\alpha = 0.05.$					Median, $\alpha = 0.05.$				
	$r = 10$									
m/Test	$D\sqrt{n}$	$V\sqrt{n}$	W^2	U^2	A^2	$D\sqrt{n}$	$V\sqrt{n}$	W^2	U^2	A^2
1	1.300	1.631	0.455	0.180	2.482	1.300	1.631	0.455	0.180	2.482
2	2.215	2.829	0.439	0.182	2.486	4.093	4.132	2.111	0.457	16.613
3	2.931	3.634	0.452	0.184	2.565	6.436	6.436	2.983	0.699	35.741

4	3.430	4.300	0.456	0.181	2.539	8.184	8.184	3.258	0.802	58.901
	$r = 20$									
1	2.215	2.822	0.439	0.182	2.486	1.300	1.631	0.455	0.180	2.482
2	2.267	2.890	0.463	0.184	2.510	5.116	5.127	3.534	0.703	26.087
3	2.940	3.717	0.462	0.182	2.560	8.459	8.459	5.536	1.241	61.456
4	3.434	4.385	0.450	0.184	2.438	11.066	11.066	6.334	1.526	102.402
	$r = 30$									
1	1.323	1.699	0.449	0.189	2.450	1.300	1.631	0.455	0.180	2.482
2	2.311	2.939	0.461	0.188	2.223	5.936	5.945	4.922	0.958	35.810
3	2.310	3.723	0.456	0.178	2.496	9.970	9.970	7.996	1.752	85.776
4	3.538	4.456	0.475	0.189	2.565	13.199	13.199	9.312	2.209	144.305
	$r = 40$									
1	1.317	1.672	0.458	0.184	2.465	1.300	1.631	0.455	0.180	2.482
2	2.296	2.926	0.459	0.182	2.466	6.562	6.562	6.243	1.171	44.986
3	2.989	3.799	0.476	0.188	2.565	11.239	11.239	10.381	2.241	109.534
4	3.540	4.436	0.474	0.185	2.546	15.034	15.034	12.285	2.892	187.042

Table (8) Efficiency values for SRS and RSS using first order statistics

H	T	Minimum, $\alpha = 0.05$.											
		$r = 10$, m			$r=20$, m			$r = 30$, m			$r = 40$, m		
		2	3	4	2	3	4	2	3	4	2	3	4
N	$D\sqrt{n}$	1.432	1.919	3.162	2.132	3.579	6.842	2.205	4.659	10.91	2.460	5.84	11.40
	$V\sqrt{n}$	1.757	2.541	3.243	2.135	3.462	4.750	2.179	3.836	5.522	2.38	4.304	6.482
	W^2	1.256	1.86	2.93	2.026	3.795	6.795	2.044	4.844	9.311	2.50	6.333	11.92
	U^2	1.490	1.898	2.571	1.690	2.724	4.0	1.940	3.373	5.164	2.092	3.934	6.211
	A^2	1.206	1.853	3.118	2.032	4	7.613	2.132	5.184	10.61	2.69	7.0	13.71
Lo	$D\sqrt{n}$	2.352	3.444	4.444	2.72	3.981	4.485	2.751	3.674	4.167	2.682	3.399	3.631
	$V\sqrt{n}$	1.683	2.163	2.528	1.495	1.953	2.294	1.406	1.749	1.946	1.312	1.542	1.677
	W^2	2.28	3.542	4.466	2.80	4.281	5.075	2.942	4.056	4.522	2.77	3.563	3.755
	U^2	1.364	1.654	1.969	1.275	1.588	1.90	1.262	1.509	1.725	1.205	1.388	1.527
	A^2	1.559	2.064	2.505	1.631	2.189	2.527	1.618	1.993	2.185	1.518	1.786	1.863
C	$D\sqrt{n}$	3.544	6.278	8.656	4.222	6.452	7.2	4.542	5.762	5.952	4.074	4.606	4.63
	$V\sqrt{n}$	2.426	4.123	5.582	2.239	3.282	3.68	1.906	2.367	2.463	1.678	1.858	1.873
	W^2	2.835	5.07	6.878	3.595	5.851	6.541	3.855	5.177	5.376	3.612	4.284	4.31
	U^2	2.124	3.572	4.772	2.108	3.108	3.517	1.937	2.419	2.538	1.734	1.953	1.972
	A^2	3.625	4.542	4.974	1.317	1.467	1.488	1.228	1.276	1.276	1.139	1.151	1.151
La	$D\sqrt{n}$	1.628	1.721	2.116	1.771	1.917	2.521	1.846	2.038	3.212	1.891	2.364	3.491
	$V\sqrt{n}$	1.771	2.114	2.343	1.88	2.36	2.56	2.107	2.786	3.411	2.203	3.078	3.688
	W^2	1.415	1.566	1.717	1.357	1.679	2.054	1.418	1.927	2.818	1.593	2.222	3.315
	U^2	1.367	1.469	1.796	1.654	2.038	2.385	1.768	2.393	3.107	2.105	2.877	3.684
	A^2	1.221	1.397	1.618	1.294	1.588	2.015	1.282	1.789	2.577	1.417	2.042	2.972

Table (9) Efficiency values for SRS and RSS using second order statistics

H	T	Median, $\alpha = 0.05$.											
		$r = 10$, m			$r = 20$, m			$r = 30$, m			$r = 40$, m		
		2	3	4	2	3	4	2	3	4	2	3	4
N	$D\sqrt{n}$	1	37	33	0.003	1.215	2.473	0.001	0.801	1	0.001	0.957	1
	$V\sqrt{n}$	1	40	34	0.003	1.224	2.471	0.001	0.803	1	0.001	0.957	1
	W^2	43	30	25	0.122	1.316	2.568	0.053	0.88	1	0.049	0.985	1
	U^2	55	40	32	0.19	0.896	2.126	0.069	0.68	1	0.088	0.885	1
	A^2	1.429	0.476	0.19	0.039	0.239	0.481	0.034	0.643	1	0.039	0.902	1
Lo	$D\sqrt{n}$	188	190	1	788	983	330	489.5	500	500	500	500	500
	$V\sqrt{n}$	189	195	1	793	984	331	326.3	333.3	333.3	166.7	166.7	166.7
	W^2	1.5	2.058	0.008	4.699	5.751	2.139	4.322	4.405	4.405	3.584	3.584	3.584
	U^2	1.069	1.28	0.005	2.275	2.922	1.09	1.951	2.024	2.024	1.645	1.645	1.645
	A^2	1.537	2.351	0.091	2.808	3.139	2.741	2.343	2.364	2.364	1.938	1.938	1.938
C	$D\sqrt{n}$	130	140	1	666	950	239	945	1000	985	994	1000	1000
	$V\sqrt{n}$	132	141	1	674	958	239	474.5	500	492.5	165.8	166.7	166.7
	W^2	1.114	1.618	0.008	4.703	6.329	1.703	4.929	5.076	5.076	3.937	3.937	3.937
	U^2	0.848	1.177	0.006	2.156	3.355	0.929	2.224	2.457	2.457	1.804	1.835	1.835
	A^2	1.204	1.715	0.002	1.443	1.534	1.521	1.278	1.282	1.282	1.152	1.152	1.152
La	$D\sqrt{n}$	1	35	38	0.034	13.45	28.14	777	996	868	951	1000	1000
	$V\sqrt{n}$	1	32	38	0.034	13.07	28.17	766	996	868	947	1000	1000
	W^2	54	42	49	3.471	30.76	56.35	15.47	17.24	17.12	17.03	17.24	17.24
	U^2	56	33	45	2.682	12.82	35	8.941	14.51	13.21	12.94	15.15	15.15
	A^2	65	73	110	67	457	807	13.06	15.55	13.2	14.95	15.38	15.38

Based on the information presented in the aforementioned tables, the following observations can be made:

1. In general, efficiency tends to increase with an expansion in the sample size, except for the A^2 test in the Cauchy case
2. Efficiency demonstrates an upward trend with an increase in set size m, except for the A^2 test in the Cauchy case.
3. Empirical Distribution Function (EDF) tests based on the i^{th} order statistic ($i=1,2,3$) exhibit higher efficiency compared to EDF tests based on the Simple Random Sampling (SRS) case ($m=1$) of the same size, except for the A^2 test in the Cauchy case.
4. Notably, in the median case (Table 9), the efficiencies of the modified tests are equivalent to their counterparts in the SRS case.

Conclusion

This paper focuses on the application of Empirical Distribution Function (EDF) tests and the chi-square Goodness-of-Fit (GOF) test under Ranked Set Sampling (RSS). A proposed method includes EDF tests and the chi-square GOF test tailored for cases where data is collected using the RSS technique. The study investigates RSS schemes that quantify specific order statistics, such as minimum, median, or maximum. Given that detecting extreme order statistics is often easier for experimenters through visual inspection, this method is particularly applicable in real-world scenarios.

The research shows an enhancement in the power of the chi-square GOF test based on RSS compared to Simple Random Sampling (SRS). A simulation study is conducted to compare the power of EDF tests and the chi-square test based on RSS against SRS. Simulation results indicate that EDF tests and the chi-square test based on minimum and maximum order statistics exhibit greater power than the chi-square test based on the median.

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